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Dupin's Cyclide as a Self-Dual Surface.

By Mabel M. Young.

Introduction.

A correlation between the points and the planes of space is determined when it is assumed that by it five arbitrary points are sent into five arbitrary planes. If the correlation sends the points of a surface into its tangent planes, the surface is called self-dual. Its order and class are then the same, and the singularities occurring in the surface considered as an envelop of planes are the dual of those found in the surface taken as a locus of points. The Kummer quartic surface is a well-known example of a surface with this property.* Another self-dual surface, not a special case of the Kummer Surface, is the Cyclide of Dupin. The determination of a group of correlations which transforms this surface into itself is the chief object of this paper.

The Determination of Four Polarities. §§ 1–3.

§ 1. *Equation of the Surface.*

Dupin's Cyclide is defined by Salmon † as a surface of the fourth order, with four double points and with a double conic in the plane at infinity. To determine its equation from this definition we take a tetrahedron of reference with vertices at the four double points of the surface. A quartic surface having a double conic in the plane at infinity may be put in the form

$$\alpha Q^2 + (\beta x)(x)Q + (\gamma x)^2(x)^2 = 0,$$

where α is a constant, (βx) an arbitrary plane, $(\gamma x)^2$ an arbitrary quadric and (x) the plane at infinity. Q is the known quadric cutting out the double conic in $(x)=0$. For simplicity, let Q contain the four lines joining the four double points. Then,

$$Q \equiv x_0x_2 + \mu x_1x_3 = 0.$$

Applying the condition that the four reference points are double points on the surface, we have the relations

$$\gamma_{00}=\gamma_{11}=\gamma_{22}=\gamma_{33}=\gamma_{01}=\gamma_{12}=\gamma_{23}=\gamma_{03}=0, \quad -2\gamma_{02}=\beta_0=\beta_2, \quad -2\gamma_{13}/\mu=\beta_1=\beta_3.$$

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* Hudson, R. W. H. T., "Kummer's Quartic Surface," § 30.

† Salmon, G., "Geometry of Three Dimensions," 4th edition, §§ 560, 567.

These values of the constants give a final form of the equation with two arbitrary constants, μ and $(\beta_1-\beta_0)/2\alpha=\lambda$.

$$(x_0x_2+\mu x_1x_3)^2+\lambda(x)\{x_0x_2(x_1+x_3)-\mu x_1x_3(x_0+x_2)\}=0. \qquad (1)$$

§ 2. *The Dual Singularities of the Surface.*

To determine a correlation which is to test the self-duality of this surface, we may make use of the fact that in a self-dual surface the singularities of the surface in points and of the surface in planes are mutually dual. We shall then look first for four double tangent planes and a double quadric cone to correspond to the four double points and the double conic which by definition are present in Dupin's Cyclide.

To find the double tangent planes, we must first consider the nature of the double points to which they are to correspond. If we write the equation of the surface according to the powers of one of its variables, we see that the tangent at a double point is a quadric cone. The double points are then "conical points" at which there are an infinite number of tangent planes enveloping a quadric cone. The dual singularity must then be a plane containing an infinity of points of the surface lying on a conic. As the double points lie by pairs on the lines $\overline{13}$ and $\overline{02}$ of the reference tetrahedron, we examine first the relation to the surface of the planes on these lines. Take the planes on $\overline{13}$, which are of the form

$$kx_0+x_2=0.$$

The tangent cones at $(0,1,0,0)$ and $(0,0,0,1)$ are, respectively,

$$\mu^2x_3^2+\lambda x_0x_2-\mu\lambda x_3(x_0+x_2)=0 \text{ and } \mu^2x_1^2+\lambda x_0x_2-\mu\lambda x_1(x_0+x_2)=0.$$

The line pairs in which a plane of the pencil cuts these cones coincide when

$$\lambda k=(\lambda-2)\pm2\sqrt{1-\lambda}.$$

There are then in the pencil two planes given by these values of k which are tangent to both of the tangent cones at these double points, and hence to the surface at these points. Moreover, the quartic curve in which each of these planes cuts the surface reduces to a repeated conic given by

$$kx_0+x_2=0, \quad \{2(kx_0^2+\mu x_1x_3)+\lambda(k-1)x_0(x_1+x_3)\}^2=0.$$

At an arbitrary point on this conic there is found to be a single tangent plane, $kx_0+x_2=0$, so that the plane is a trope. Similarly, there are among the planes on $\overline{02}$, $lx_1+x_3=0$, two tropes given by

$$\lambda l=(\lambda+2\mu)\pm2\sqrt{\mu\lambda+\mu^2}.$$

We have then found four planes which have the properties required for the planes dual to the four double points. These four tropes are

$$k_1x_0+x_2=0,\ k_2x_0+x_2=0,\ l_1x_1+x_3=0,\ l_2x_1+x_3=0, \tag{2}$$

when k_1, k_2, l_1, l_2 are the two sets of values for k and l already found.

The double conic of the surface is characterized by the fact that at each point the surface has two tangent planes. The dual of the conic must then be a quadric cone, every plane of which meets the surface in two points. Further, the tangent planes of the surface at points of the conic meet the plane $(x)=0$ in lines tangent to the conic. Hence, dually, the two contacts of each plane of the cone lie on a line passing through a fixed point, the vertex of the cone. If we can find such a cone of double tangent lines, we may consider its vertex as a point dually related to the plane of the double conic.

Before finding the equation of the cone and its vertex, it will be useful to notice another locus on its surface. Since the elements are double tangent lines of the surface, the points of contact may be given by a binary quartic with two pairs of equal roots. Let 0 and ∞ be the parameters of the points of contact on any element; then the binary quartic is $x_1^2y_2^2=0$. If z_1+z_2 is the vertex, its first polar as to the quartic is $(z_1y_2+z_2y_1)x_1y_2=0$; its third $z_1z_2(z_1y_2+x_1z_2)=0$. These have the common point $z_1y_2+x_1z_2$. As this relation occurs on every element, we see that the first and third polar surfaces of the vertex as to (1) meet in a conic on the cone and in a line.

The equation of the cone is most easily found if we rewrite the equation of the surface in terms of the parameter of the points $x_i+\lambda y_i$ in which a line on x_i and y_i meets it. The equation becomes

$$p_4+p_3\gamma+p_2\gamma^2+p_1\gamma^3+p\gamma^4=0, \tag{3}$$

where $p=$equation of surface, and $i!\, p_i=i$-th polar of surface as to a point z. If the line cutting the surface is an element of a cone of double tangent lines, it meets the surface in two points only, and (3) has two pairs of equal roots. Take z at the vertex. For simplicity set $p_1\equiv0$. Then $p_3\equiv0$ also, and the equation becomes

$$p_4+p_2\gamma^2+p\gamma^4=0. \tag{4}$$

The condition for equal roots is now the vanishing of $4p_4p-p_2^2$. This expression set equal to zero gives an equation in x and z which vanishes when x is a point on a double tangent through z. It must then for some special value of z be the equation of the double tangent cone from z.

The vertex is the point $(\mu, 1, \mu, 1)$. This fact is most easily established by verifying that for this value of z the discriminant $4p_4p-p_2^2\equiv0$. The polars, found for this point, are

$$p_1=2[(x)(\mu Q+\lambda R)+\lambda\nu\{x_0x_2(x_1+x_3)-\mu x_1x_3(x_0+x_2)\}],$$
$$p_2=2\nu(\mu Q+\lambda R),\quad p_3=2\mu\nu\{\sigma(x_0+x_2)+\mu\rho(x_1+x_3)\},\quad p_4=\mu^2\nu^2.$$
$$Q\equiv x_0x_2+\mu x_1x_3;\quad R\equiv x_0x_2-\mu^2x_1x_3;\quad \mu+1\equiv\nu;\quad 1-\lambda\equiv\rho;\quad \mu+\lambda\equiv\sigma.$$

Substituting in the discriminant these values of p_i we have, as the equation of the cone

$$4\nu(\mu Q+\lambda R)-\{\sigma(x_0+x_2)+\mu\rho(x_1+x_3)\}^2=0. \tag{5}$$

But a surface on the intersection of $p_1=0$ and $p_3=0$ is easily seen to be $\mu Q+\lambda R=0$; hence $\mu Q+\lambda R$ is known to be identically zero when the polar plane of $(\mu, 1, \mu, 1)$ as to the surface is zero. Therefore the discriminant of (4) vanishes for this point and we see that the cone of double tangent lines, the dual of the double conic in $(x)=0$, is a quadric cone with the vertex $(\mu, 1, \mu, 1)$. This point is accordingly the dual of $(x)=0$.

We have now five pairs of points and planes associated with Dupin's Cyclide, and dually related to each other,—the four double points corresponding to the four tropes, and the vertex of the cone of double tangent lines corresponding to the plane at infinity. These are sufficient to determine the correlation which we wish to consider.

§ 3. *The Equations of the Transformation.*

Let the correlations be given by the transformation

$$\rho u_i=\alpha_{i0}x_0+\alpha_{i1}x_1+\alpha_{i2}x_2+\alpha_{i3}x_3, \tag{6}$$

and assume $\alpha_{ik}=\alpha_{ki}$. This will give rise to a quadric with respect to which the corresponding point and plane will be pole and polar. In pairing the points and planes a point will not then, in general, be on its corresponding plane. Let the points on $\overline{13}$ and $\overline{02}$ be paired with the planes on $\overline{02}$ and $\overline{13}$ respectively:

Point.	*Plane.*
(1, 0, 0, 0)	$k_1x_0+x_2=\sigma$
(0, 1, 0, 0)	$l_1x_1+x_3=\sigma$
(0, 0, 1, 0)	$k_2x_0+x_2=\sigma$
(0, 0, 0, 1)	$l_2x_1+x_3=\sigma$

If we regard the equation of each plane as the result of applying to the corresponding point the transformation (6), we determine six relations among the coefficients α_{ik}:

$$\alpha_{03}=\alpha_{01}=\alpha_{12}=\alpha_{23}=0,\quad \alpha_{00}=\alpha_{22},\quad \alpha_{11}=\alpha_{33}.$$

The further requirement that by the same transformation $(\mu, 1, \mu, 1)$ goes into $(x)=0$ fixes the ratios

$$\frac{\alpha_{33}}{\alpha_{22}}=\frac{\alpha_{11}}{\alpha_{00}}=\frac{(1+k_1)\mu l_1}{(1+l_1)k_1};\quad \frac{\alpha_{13}}{\alpha_{00}}=\frac{\mu(1+k_1)}{k_1(1+l_1)}.$$

But such a transformation is a polarity with respect to a quadric $(\alpha x)^2=0$, the coefficients of which are the constants of the transformation. Hence, setting $k_i+1=n_i$ and $l_i+1=m_i$, we have the quadric

$$Q\equiv(\alpha x)^2\equiv k_1m_1x_0^2+\mu n_1l_1x_1^2+\mu m_1x_2^2+\mu l_1n_1x_3^2+2m_1x_0x_2+2\mu n_1x_1x_3=0.$$

As k_i and l_i have each two values connected by the relation $k_1k_2=l_1l_2=1$, we may immediately derive from $(\alpha x)^2=0$ three other quadrics which meet the conditions, and give rise to three new polarities. Our result may then be stated as a

LEMMA. *There are four transformations, polarites as to four quadrics, which send the point singularities of Dupin's Cyclide into their dual forms.*

The theorem that the entire surface is self-dual under the same polarities will be proved by another method in § 8.

The initial hypothesis about the transformation

$$\rho u_i=\sum_{k=0}^{k=3}\alpha_{ik}x_k$$

was that $\alpha_{ik}=\alpha_{ki}$. If we had assumed $\alpha_{ik}=-\alpha_{ki}$, the transformation would have led to a linear complex, instead of a quadric. But in such a case a point and its corresponding plane are incident. As this is geometrically impossible for $(\mu, 1, \mu, 1)$ and its corresponding plane $(x)=0$, it is evident that we cannot show that these five pairs of points and planes, and much less the points and planes of the surface, belong to a null-system.

A STUDY OF COVARIANT FORMS. §§ 4–7.

§ 4. *Relations Among the Quadrics Q_i.*

Before studying these polarities further, it is perhaps worth while to examine in some detail the geometry of certain covariant forms closely associated with the surface. Among these are the four quadrics Q_i found in § 3, which are closely connected. Because $k_1k_2=l_1l_2=1$, it is possible to write all the quadrics in terms of k_1 and l_1, and in this form it is readily seen that a linear relation exists among them. The equations * are

$$\left.\begin{aligned}Q_1&\equiv m(kx_0^2+kx_2^2+2x_0x_2)+\mu n(lx_1^2+lx_3^2+2x_1x_3)=0,\\ lQ_2&\equiv m(kx_0^2+kx_2^2+2x_0x_2)+\mu n(x_1^2+x_3^2+2lx_1x_3)=0,\\ kQ_3&\equiv m(x_0^2+x_2^2+2kx_0x_2)+\mu n(lx_1^2+lx_3^2+2x_1x_3)=0,\\ lkQ_4&\equiv m(x_0^2+x_2^2+2kx_0x_2)+\mu n(x_1^2+x_2^2+2lx_1x_3)=0.\end{aligned}\right\}\quad(7)$$

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* A transformation of the reference tetrahedron to the self-conjugate tetrahedron of the quadrics gives all forms related to the quadrics more simply. Other forms in frequent use become, however, so complicated and unsymmetrical that this desirable transformation is not made.

That a linear relation exists is known because the Jacobian of the four quadrics vanishes. To determine this relation we combine the equations in pairs in all possible ways.

$$Q - lQ_2 = \mu nh(x_1 - x_3)^2, \quad kQ_3 - klQ_4 = \mu nh(x_1 - x_3)^2. \tag{8}$$

$$Q_1 - kQ_3 = mr(x_0 - x_2)^2, \quad lQ_2 - lkQ_4 = mr(x_0 - x_2)^2. \tag{9}$$

For each of these pencils of quadrics the invariant Φ[*] $\equiv 0$,

$$\left.\begin{aligned} &lQ_2 - kQ_3 = mr\{(x_0 - x_2)^2 - \mu nh(x_1 - x_3)^2\}, \\ &Q_1 - klQ_4 = mr\{(x_0 - x_2)^2 + \mu nh(x_1 - x_3)^2\}, \qquad h \equiv l - 1;\ r \equiv k - 1. \end{aligned}\right\} \tag{10}$$

For these two quadrics the invariants Θ^* and $\Theta' \equiv 0$.

$$lQ_2 + kQ_3 = Q_1 + klQ_4 = mn\{(x_0 + x_2)^2 + \mu(x_1 + x_3)^2\}. \tag{11}$$

From (8) or (10) we see that the linear relation connecting the quadrics is

$$Q_1 - kQ_3 - lQ_2 + klQ_4 \equiv 0.$$

From (8) we see also that the pairs Q_1 and Q_2, Q_3 and Q_4 have a repeated conic in the plane $x_1 - x_3 = 0$, and are accordingly tangent along the conic in which they meet it. Equations (9) give the same relation for Q_1 and Q_3, Q_2 and Q_4, which have their common curve in $x_0 - x_2 = 0$. Since each quadric contains a fixed conic in $x_0 - x_2 = 0$ and in $x_1 - x_3 = 0$, the intersection of these planes meets all four quadrics at the same two points, and at these points the quadrics have double tangency. From (10) we learn that Q_1 and Q_4, Q_2 and Q_3 meet also in other conics in the planes $x_1 + x_3 = 0$, $x_0 + x_2 = 0$, and, accordingly, have double contact when the line

$$x_1 + x_3 = 0, \quad x_0 + x_2 = 0$$

meets each pair. Each quadric has then "ring contact" with two others and double contact with a third. At two points on one line all four quadrics have double contact. On a second line they have double contact by pairs at two sets of points which are readily seen to form a harmonic set.

§ 5. *Certain Tangent Cones.*

The enveloping cone to the surface from a point without is of order 12, since it consists of tangent lines drawn to the intersection of the surface and its first polar. Consider the cone from $(\mu, 1, \mu, 1)$. The first polar of this point as to the surface is

$$(x)(\sigma x_0 x_2 + \mu^2 \rho x_1 x_3) + \lambda \nu x_0 x_2 \overline{x_1 + x_3} - \mu x_1 x_3 \overline{x_0 + x_2} = 0.$$

This meets the plane at infinity in the conic

$$x_0 x_2 + \mu x_1 x_3 = 0, \quad (x) = 0.$$

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[*] Salmon, G., "Analytic Geometry of Three Dimensions," 5th edition, Vol. I, Chap. 9.

The polar cubic also contains the double points of the surface and the lines joining them, along which it is tangent to the cyclide. The complete cone to the surface must then contain a cone to the double conic, counted twice; the four planes tangent to the surface along its four lines, and the cone of double lines already found. The planes are

$$1)\ x_0-\mu x_1=0;\ 2)\ x_0-\mu x_3=0;\ 3)\ x_2-\mu x_1=0;\ 4)\ x_2-\mu x_3=0. \qquad (12)$$

The cone to the double conic is

$$4\nu(x_0x_2+\mu x_1x_3)-\mu(x)^2=0, \qquad (13)$$

The lines along which planes (12) are tangent to the surface meet the double cone in four points whose coordinates are

$$(0,0,1,-1);\ (0,1,-1,0);\ (1,0,0,-1);\ (1,-1,0,0). \qquad (14)$$

They meet the cone of double lines in points

$$(0,0,\mu\rho,-\sigma);\ (0,-\sigma,\mu\rho,0);\ (\mu\rho,0,0,-\sigma);\ (\mu\rho,-\sigma,0,0).$$

Planes (12) are thus tangent to both cones. Hence, we see that the enveloping cone from $(\mu, 1, \mu, 1)$ to the surface consists of two quadric cones and their four common tangent planes.

At the points (14) the double conic is met also by the four tangent cones on the double points of the surface. These cones are

$$x_2^2+\lambda x_1x_3+\lambda x_2x_3-\mu\lambda x_1x_3=0,\quad \mu^2x_3^2+\lambda x_0x_2-\mu\lambda x_0x_3-\mu\lambda x_2x_3=0,$$
$$x_0^2+\lambda x_0x_1+\lambda x_0x_3-\mu\lambda x_1x_3=0,\quad \mu^2x_1^2+\lambda x_0x_2-\mu\lambda x_0x_1-\mu\lambda x_1x_2=0.$$

Each contains two lines of the surface. The pairs of cones with vertices on the diagonals $\overline{02}$, $\overline{13}$, which are touched by the tropes, meet each other on the surface only at the two remaining double points. Any two cones, except these pairs, have one line in common, along which they are touched by one of the four tangent planes (12). In $(x)=0$ we have thus at each of the four points (14) two cones tangent to each other and to the double conic. Each cone has double contact with the conic, but single contact only with each of the two cones which it meets.

§ 6. *The Four Pinch Points and their Duals.*

In general, at every point y of the double conic the surface has two distinct tangent planes. These are the factors of the polar quadric of y,

$$2\{x_0y_2+x_2y_0+\mu(x_1y_3+x_3y_1)\}$$
$$-\{\lambda(y_0+y_2)\pm\sqrt{\lambda^2(y_0+y_2)^2-4\lambda y_0y_2}\}(x)=0. \qquad (15)$$

When, however, the expression under the radical is zero, the two planes coincide. The condition for this is

$$\lambda y_2 = -\{(\lambda-2) \pm 2\sqrt{1-\lambda}\} y_0,$$

with the further necessary relation

$$\lambda y_3 = -\{(\lambda+2\mu) \pm 2\sqrt{\mu^2+\lambda\mu}\} y_1.$$

The constants involved are, respectively, the values of k_i and l_i in the equations of the double tangent planes (2), so that the ratios become

$$y_2/y_0 = -k_i, \quad y_3/y_1 = -l_i.$$

Each of these four points satisfies two tropes, $(x)=0$, and the cone of double tangent lines, as well as the double conic. They are then the vertices of the quadrilateral cut out by the tropes in $(x)=0$ and inscribed in the double conic at its intersections with the cone of double lines. The coordinates of the four points, called "pinch points," are

$$(-h_i,\ r_i,\ k_i h_i,\ -l_i r_i), \qquad (i=1,2).$$

The tangent planes to the surface at these points are

$$m_i\rho(k_i x_0 + x_2) - n_i\sigma(l_i x_1 + x_3) = 0.$$

The fact that the tropes pass through these points enables us to state the relations between the singular conics in these planes. As we have already seen, the conics in a pair of tropes on a diagonal line meet the double points on this line. At each pinch point two tropes meet which are not on a diagonal, and hence the corresponding singular conics meet also. Each conic accordingly meets one other conic at two double points of the surface and a second at a pinch point. These remarkable points are thus seen to have the following properties. They are the only points of the double conic at which the surface has a single tangent plane and they are each found on the double conic, the cone of double lines, two tropes and two singular conics.

The duals of the pinch points are planes already found. At each point the single tangent plane of the surface cuts $(x)=0$ in a line tangent to the double conic. Dually on the cone of double lines must be an element meeting the surface in two coincident points. The tangent plane on this line meets the surface in four coincident points. Moreover, the pinch points are common to two singular conics not lying in tropes on a diagonal. The corresponding planes must then be common planes of two cones of the surface at double points not on a diagonal. Such planes are the singular planes (12), and the coincident points of contact are the points (14), which are on the conic cut from the cone of double lines by the polar plane of its vertex. The planes of the surface dual to the pinch points are therefore the singular tangent planes of the surface.

§ 7. *A Quartic Curve on the Surface.*

The only space curve on the surface which arises in the preceding discussion is the locus of the points of contact of the cone of double lines and the surface. This curve is also on the polar of the surface as to $(\mu, 1, \mu, 1)$. Since the further intersection of this cubic and the cone is a conic (§ 2), the locus of contacts is a quartic curve* cut out by a pencil of quadrics.†

By combining the equations of the surface, the first polar as to $(\mu, 1, \mu, 1)$ and the cone, we obtain a quartic surface which must contain all points common to the three surfaces. This is

$$[2\nu(x_0x_2+\mu x_1x_3)-(x)\{\sigma(x_0+x_2)+\mu\rho(x_1+x_3)\}]$$
$$[2\nu(x_0x_2+\mu x_1x_3)+(x)\{\sigma(x_0+x_2)+\mu\rho(x_1+x_3)\}]=0. \qquad (16)$$

To decide which of these quadrics cuts out the curve under discussion, we may put the equation of the surface in a form which will show its relation to the cone of double lines. Such a form is

$$S=Q_1^2+C_1Q_2,$$

where S is the surface, C_1 the cone of double lines, Q_1 a quadric through the contacts and Q_2 an arbitrary quadric. This equation expresses that S is tangent to C_1, where C_1 meets Q_1. Hence, that one of the two quadrics (16) passes through the curve, the square of which subtracted from the surface gives the cone multiplied by some quadric. Q_1 is found to be

$$2\nu(x_0x_2+\mu x_1x_3)-(x)\{\sigma(x_0+x_2)+\mu\rho(x_1+x_3)\}=0. \qquad (17)$$

The pencil of quadrics which cuts out the quartic curve of contacts on the cone of double lines is therefore given by linear combinations of (17) and (5).

MAPPING OF THE SURFACE. §§ 8–11.

§ 8. *Self-duality of Entire Surface.*

By the principles of mapping it is possible to show that the four polarities of § 3 do, in fact, send every point of Dupin's Cyclide into one of its planes. That the surface may be mapped follows at once from the fact that it is rational This is seen by considering the intersections with the surface of a line which moves always touching the double conic and one line of the surface. Three of its four intersections are thus accounted for; the fourth is variable. If the cutting line also meets some arbitrary plane, this point and the variable intersection with the surface are always in 1, 1 correspondence. The surface thus meets the test for rationality and may be mapped.

In mapping a surface of given order, the curves which map plane sections must pass through a sufficient number of base points so that the free intersections of any two shall be equal to the order of the surface. To map a quartic, therefore, by means of general cubics, we need five base points. We have in this way ∞^4 cubics. Now, as the ∞^3 planes of space are linear functions of the four independent planes, they must be mapped by ∞^3 cubics on the five base points, which are linear functions of the four fundamental cubics. All such cubics must accordingly meet one further condition. Let this be apolarity to a general line cubic, $(a\xi)^3=0$. Plane sections of the surface are then mapped by a linear system of cubics in the plane.

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* Salmon, G., "Geometry of Three Dimensions," 5th edition, Vol. I, p. 363.

† The breaking up of this curve into two conics is the condition that the surface take the special form of the "Anchor Ring."

Before deriving specific transformations for the mapping, we must take account of the fact that the surface has four double points. This leads to a special arrangement of the base points. In general, it is assumed that no three shall lie on a straight line, but when the surface has a double point, this restriction is removed. This is because each of the net of planes which may be passed through a double point on the surface cuts out a rational section, and must be mapped by a cubic of the same genus. There are thus among the ∞^3 cubics on five points and apolar to $(a\xi)^3=0$ four nets of rational cubics. Two such nets are: 1) cubics consisting of a line on three points and a net of conics on the two remaining apolar to the polar conic of the line as to $(a\xi)^3=0$; 2) nets of proper rational cubics with the double point at a base point. Two such independent arrangements of three points on a line, as in case 1, may be made. Case 2 arises when two of the points on a line approach coincidence. Hence of the ∞^3 cubics through these points, a net will have a double point. Two such nets are possible. We have thus accounted for peculiarities due to the presence of four double points on the surface, if we take the five base points in two coincident pairs on two lines meeting at the fifth point.

Let these points be the vertices of the reference triangle in the mapping plane, and let the coincident pairs be $(0, 1, 0)$ and a point consecutive to it on $u_2=0$, $(0, 0, 1)$ and a consecutive point on $u_1=0$. A cubic curve through these points and apolar to $(a\xi)^3=0$ is of the form

$$(\alpha u)^3 \equiv \alpha_{001}u_0^2u_1+\alpha_{002}u_0^2u_2+\alpha_{112}u_1^2u_2+\alpha_{122}u_1u_2^2 \\ +(a\alpha_{001}+b\alpha_{002}+c\alpha_{112}+d\alpha_{122})u_0u_1u_2=0.$$

Making the first four coefficients zero in turn, we have four cubics meeting the required conditions

$$f_0\equiv u_0u_1u_2+\beta_0u_0^2u_1=px_0\,, \quad f_1\equiv u_0u_1u_2+\beta_1u_0^2u_2=qx_1\,,$$
$$f_2\equiv u_0u_1u_2+\beta_2u_1u_2^2=px_2\,, \quad f_3\equiv u_0u_1u_2+\beta_3u_1^2u_2=qx_3\,.$$

The elimination of u from these equations gives the surface in form (1), if certain obvious substitutions are made for p and q. The transformation then takes the final form

$$\left.\begin{aligned} f_0 &\equiv x_0 = u_0u_1u_2 - \rho u_0^2u_1/\lambda\beta_2\,, \\ f_1 &\equiv \mu x_1 = u_0u_1u_2 + \sigma u_0^2u_2/\lambda\beta_3\,, \\ f_2 &\equiv x_2 = u_0u_1u_2 + \beta_2u_1u_2^2\,, \\ f_3 &\equiv \mu x_3 = u_0u_1u_2 + \beta_3u_1^2u_2\,. \end{aligned}\right\} \quad (18)$$

We can now show directly that Dupin's Cyclide is self-dual under the polarities of § 3. Any plane cuts the surface in a quartic curve with two double points which maps into a non-rational cubic. Since the point of contact of a tangent plane is a double point on the surface, each tangent plane cuts the quartic surface in a rational quartic which maps into a rational cubic on five points. The ∞^2 tangent planes thus give ∞^2 rational cubics. Consequently, if a point y on the surface has as polar plane with respect to one of the quadrics Q_i (7) a tangent plane of the surface, the map of the section will be rational. Assume such a tangent plane as the polar plane of y. Then,

$$\left.\begin{aligned} P_1(x) &\equiv m(ky_0+y_2)x_0+\mu n(ly_1+y_3)x_1+m(y_0+ky_2)x_2+\mu n(y_1+ly_3)x_3=0, \\ P_1(f) &\equiv m(ky_0+y_2)\{u_0u_1u_2-\rho u_0^2u_1/\lambda\beta_2\}+n(ly_1+y_3)\{u_0u_1u_2+\sigma u_0^2u_2/\lambda\beta_3\} \\ &\quad +m(y_0+ky_2)\{u_0u_1u_2+\beta_2u_1u_2^2\}+n(y_1+ly_3)\{u_0u_1u_2+\beta_3u_1^2u_2\}=0. \end{aligned}\right\} \quad (19)$$

This cubic is of the form, in Salmon's coefficients,*

$$a_1u_2^2u_1+u_2(b_2u_1^2+b_0u_0^2+2b_1u_0u_1)+c_1u_0^2u_1=0.$$

If this cubic is rational, its discriminant will vanish. The discriminant is

$$\Delta \equiv 27a_1c_1b_0b_2[b_1^4-2b_1^2(b_0b_2+a_1c_1)+(b_0b_2-a_1c_1)^2].$$

Substituting from (19) the values of the constants and using the relations† which exist among k, l, λ, μ, this becomes

$$\begin{aligned} \Delta \equiv\ & (y_0+ky_2)^2(ky_0+y_2)^2(y_1+ly_3)^2(ly_1+y_3)^2 \\ & [(y_0y_2+\mu y_1y_3)^2+\lambda(y)\{y_0y_2(y_1+y_3)-\mu y_1y_3(y_0+y_2)\}]. \end{aligned}$$

This expresses that the discriminant vanishes identically when 1) y is a point on a singular conic and when 2) y is any point on the original surface (1). As the same result is obtained by using the polar plane of y as to each of the remaining quadrics, we may state the

THEOREM. *Dupin's Cyclide is a surface self-dual under four polarities.*

§ 9. *Plane Equation of Surface.*

We may note here that the discriminant of the map equation of a tangent plane gives the equation of the surface in planes. Substituting for y in the discriminant its coordinates as the pole of a plane p, we have the equation of the surface in the terms of η, the coordinates of the plane,

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* Salmon, G., "Higher Plane Curves," 3d edition, p. 194.

† $\sqrt{\lambda}\,l_1 = \sqrt{\lambda+\mu} + \sqrt{\mu}$, $\qquad \sqrt{\lambda}\,l_2 = \sqrt{\lambda+\mu} - \sqrt{\mu}$

$\sqrt{\lambda}\,k_1 = -\sqrt{1-\lambda} + \sqrt{1}$, $\qquad \sqrt{\lambda}\,k_2 = -\sqrt{1-\lambda} - \sqrt{1}$,

$\lambda(k-1)^2 = -4k$, $\qquad \lambda(l-1)^2 = 4\mu l$,

$\lambda(k+1)^2 = -4(1-\lambda)k$, $\qquad \lambda(l+1)^2 = 4(\mu+\lambda)l$,

$\lambda(k^2+1) = 2k(\lambda-2)$, $\qquad \lambda(l^2+1) = 2l(2\mu+\lambda)$.

$$y_0=\mu h(k\eta_0-\eta_2), \qquad y_1=r(l\eta_1-\eta_3),$$
$$y_2=\mu h(k\eta_2-\eta_0), \qquad y_3=r(l\eta_3-\eta_1),$$
$$S\equiv[\mu^2(4\eta_0\eta_2+\lambda\overline{\eta_0-\eta_2}^2+4\mu\eta_1\eta_3-\lambda\overline{\eta_1-\eta_3}^2]^2+4\mu\lambda(\mu\,\overline{\eta_0+\eta_2}+\eta_1+\eta_3)$$
$$[\overline{\mu\eta_1+\eta_3}\{4\eta_0\eta_2+\lambda(\eta_0-\eta_2)^2\}-(\eta_0+\eta_2)\{4\mu\eta_1\eta_3-\lambda(\eta_1-\eta_3)^2\}]=0.$$

§ 10. *The Mapping of Plane Sections.*

In general, the map of a plane section of this surface is a quartic curve of genus 1, which is in 1, 1 correspondence with a cubic of the same genus in the mapping plane. There are, however, four classes of plane sections, beside those made by the tangent planes, which are rational and accordingly map into rational cubics. A single section also exists which is not in 1, 1 correspondence with its mapping curve. We shall take up separately the maps of these sections.

Two sets of rational sections are those made by planes through the double points and the lines of the surface. Through each double point is a net of planes cutting sections which have this point as a third double point.

Double Point.	*Map of Planes through Point.*
η_0	$u_2=0$; a net of conics tangent to $u_1=0$ at $(0,0,1)$.
η_1	$u_1=0$; a net of conics tangent to $u_2=0$ at $(0,1,0)$.
η_2	A net of cubics with double point at $(0, 0, 1)$ and tangent to $u_2=0$ at $(0, 1, 0)$.
η_3	A net of cubics with double point at $(0, 1, 0)$ and tangent to $u_1=0$ at $(0, 0, 1)$.

By considering the part of the mapping curve common to all curves of the net, we determine the maps of the double points.

Double Point:	η_0,	η_1,	η_2,	η_3.
Map:	$u_2=0$,	$u_1=0$,	$(0, 0, 1)$,	$(0, 1, 0)$.

Sections of the surface by pencils of planes through a line on the surface break into the line and a pencil of rational cubics with double points on the double conic.

Line.	*Map of Planes through Line.*
$\overline{\eta_0\eta_1}$	$u_1=0$; $u_2=0$; a pencil of lines through $(1, 0, 0)$.
$\overline{\eta_2\eta_3}$	$u_0=0$; a pencil of conics on three reference points.
$\overline{\eta_1\eta_2}$	$u_1=0$; a pencil of conics tangent to $u_2=0$ at $(0, 1, 0)$ and passing through $(0, 0, 1)$.
$\overline{\eta_0\eta_3}$	$u_2=0$; a pencil of conics tangent to $u_1=0$ at $(0, 0, 1)$ and passing through $(0, 1, 0)$.

(20)

The maps of the lines of the surface are thus known.

$$\begin{array}{lcccc} \textit{Line}: & \overline{\eta_0\eta_1}, & \overline{\eta_2\eta_3}, & \overline{\eta_1\eta_2}, & \overline{\eta_0\eta_3}. \\ \textit{Map}: & u_1u_2=0, & u_0=0, & u_1=0, & u_2=0. \end{array}$$

In each plane pencil is one plane tangent to the surface along the entire line and hence cutting the surface in a conic and the line counted twice.

$$\left.\begin{array}{ccr} \textit{Line.} & \textit{Singular Plane.} & \textit{Map.} \\ \overline{\eta_0\eta_1} & x_2-\mu x_3=0 & u_1u_2[\beta_2u_2-\beta_3u_1]=0 \\ \overline{\eta_2\eta_3} & x_0-\mu x_1=0 & u_0^2[\rho\beta_3u_1+\beta_2\nu u_2]=0 \\ \overline{\eta_1\eta_2} & x_0-\mu x_3=0 & u_1[\rho u_0^2+\beta_3u_1u_2]=0 \\ \overline{\eta_0\eta_3} & x_2-\mu x_1=0 & u_2[\sigma u_0^2-\lambda\beta_2\beta_3u_1u_2]=0 \end{array}\right\} \quad (21)$$

Before considering the remaining classes of rational plane sections of the surface, it is of advantage to map the double conic in the plane at infinity. Its map equation is

$$\mu\lambda\beta_2\beta_3C\equiv\sigma\beta_2u_0^2u_2-\rho\mu\beta_3u_0^2u_1+\mu\lambda\beta_2^2\beta_3u_1u_2^2+\lambda\beta_2\beta_3^2u_1^2u_2+2\nu\lambda\beta_2\beta_3u_0u_1u_2=0. \quad (22)$$

The conic is unique in that it is in 1—2 correspondence with the cubic C. The two points of the conic which are double points on every plane section appear in the mapping plane as four common points of the maps of the section and the conic. If a pencil of planes is taken on a secant of the double conic, the corresponding pencil of cubics in the mapping plane will be determined by nine points on C,—the five base points, which are on the maps of all plane sections, and the four points which map the intersection of the conic and secant. If, however, the base of the pencil of planes becomes a tangent of the conic, the four variable points on C become two pairs of consecutive points at which each cubic of the pencil has double contact with C. Since every fixed point through which a pencil of curves passes with a fixed tangent is the double point of one curve of the pencil, we find in this pencil of non-rational cubics two rational curves with double points, respectively, at one or the other of the two points on C. These two cubics map the sections of the surface by the two tangent planes in the plane pencil. At the four points in which the double conic is met by a line on the surface, one of these two tangent planes is a singular plane containing the line. The map of this section accordingly degenerates. At one of the two points on C mapping such a point, we have a double point of a proper rational cubic; at the other a double point of a degenerate cubic. Four exceptional points on the double conic must, moreover, be noted. These are the pinch points, which are mapped by a single point each. The pencil of planes on a tangent to the double conic at one of these points is mapped by a pencil of cubics having contact of the second order with C at the corresponding point. The one rational curve in the pencil has the tangent of C as one tangent at the double point. It maps the single tangent plane of the surface at the pinch point.

If now we consider in relation to C the pencil of planes on a line of the surface (20), the map of each section will contain the two fixed intersections with C which map the common point of line and double conic, while the variable points will map the other double point of the section. The two fixed points will be those mapping the contact of the singular plane in the pencil. In the case of planes on $\overline{\gamma_0\gamma_1}$, two intersections with C of the variable line must map the variable double point of the section. The third intersection and a fourth point must be fixed, and map the fixed intersection of line and conic. Hence, the variable lines pass through a fixed point and form a pencil. We determine this point from a line in the map of the singular plane in the pencil, $\beta_2u_2-\beta_3u_1=0$. This intersects the variable line and touches C at $(-\beta_2\beta_3, \beta_2, \beta_3)$, and meets C again at $(1, 0, 0)$. These two points are accordingly the map of the fixed double point, and $(-\beta_2\beta_3, \beta_2, \beta_3)$ is the vertex of the pencil.

The discussion of the planes through the three other lines gives the same results from a more general point of view. The planes on $\overline{\gamma_2\gamma_3}$, for example, are mapped by $u_0=0$ and conics on three reference points. The map of the singular plane is

$$u_0^2(\rho\beta_3u_1+\sigma\beta_2u_2)=0.$$

Now $u_0=0$ meets C in $(0, 1, 0)$, $(0, 0, 1)$ and $(0, -\mu\beta_2, \beta_3)$. The line

$$\rho\beta_3u_1+\sigma\beta_2u_2=0$$

touches C in $(\lambda\beta_2\beta_3, -\beta_2\sigma, \beta_3\rho)$ and has $(1, 0, 0)$ as residual intersection. Of these five points all but $(0, -\mu\beta_2, \beta_3)$ are on all conics of the system. These points, however, are found only on the map of the singular plane, and map its contact with the double conic. The cubics in planes on $\overline{\gamma_2\gamma_3}$ are then mapped by a pencil of conics on four points of C. The two remaining intersections of each conic map the double point of the corresponding section, and these points, by the geometry of the plane cubic, lie on a line through a fixed point of C. This is readily seen to be $(-\beta_2\beta_3, \beta_2, \beta_3)$. Exactly similar results are obtained for the planes on the two remaining lines.

We have thus verified for Dupin's Cyclide the properties stated by Clebsch* for the general quartic surface with a double conic; that planes on a line of the surface are mapped by 1) the map of the line and 2) a pencil of curves with base points on the map of the double conic and meeting this curve also in pairs of points which map the single points of the double conic. These point pairs determine a central involution with vertex on the map of the conic.

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* Borchardt, *Journal für Mathematik*, Vol. LXIX (1868), p. 147.

Sections made by planes on the diagonal lines of the reference tetrahedron are two conics. The map equations give, for planes on $\overline{\eta_1\eta_3}$, a pencil of degenerate conics with double point at $(0, 1, 0)$, and $u_1=0$; for planes on $\overline{\eta_0\eta_2}$ a similar pencil of conics with double point at $(0, 0, 1)$, and $u_2=0$. Special cases of these planes are the tropes which touch the surface along conics counted twice.

$$\left.\begin{array}{ll} \textit{Trope.} & \textit{Map.} \\ k_i x_0 + x_2 = 0 & u_1(k_i+1)(u_0+2\beta_2 u_2)^2 = 0 \\ l_i x_1 + x_3 = 0 & u_2(l_i+1)(u_0+2\beta_3 u_1)^2 = 0 \end{array}\right\} \quad (23)$$

The points where the tropes meet the double conic are mapped by the coincident point pairs in which the four tangents from $(-\beta_2\beta_3, \beta_2, \beta_3)$ to C meet the curve. From the coordinates of these points we find that the maps of the tropes pass through them by pairs just as the tropes pass by pairs through the pinch points on the double conic. These four points, which map the pinch points, are

$$\{-2\beta_2\beta_3, \ \beta_2(l_i+1), \ \beta_3(k_i+1)\} \qquad (i=1, 2).$$

A tangent plane to the cone of double lines also cuts the surface in two conics. The map of a section is a cubic for which the invariants* S and T both vanish. Hence, the mapping cubic has a cusp.

We have thus considered in detail the map of the double conic, and the maps of the five sets of planes giving rise to rational sections. These groups of plane sections and the general forms of their maps are as follows:

1. A double infinity of simple tangent planes. These map into ∞^2 proper rational cubics.

2. Four nets of planes through the double points. These map into four nets of rational cubics, two of which are degenerate.

3. Four pencils of planes on the lines of the surface. These map into four pencils of degenerate cubics consisting of three lines or a line and a conic.

4. Two pencils of planes on the diagonals of the reference tetrahedron. These map into two pencils of degenerate cubics consisting of three lines.

5. A single infinity of planes tangent to the cone of double lines. These map into ∞^1 rational cubics with a cusp.

The section by the plane at infinity, the double conic, has been shown to be in 1, 2 correspondence with a non-rational cubic of the mapping plane, and the relation between the two curves has been found to be the same as that between the corresponding curves on the most general quartic surface with a double conic.

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* Salmon, G., "Higher Plane Curves", 3d edition, p. 210.

§ 11. *The Map of a Quartic Curve.*

The quartic curve which is the locus of contacts of elements of the double cone is cut out by a quadric (§ 7). Substituting $x_i=f_i$, we have the map equation of the curve. This falls into two factors. one of which is the map of the double conic, the other

$$\beta_3\sigma(\rho u_0^2u_1-\lambda\beta_2^2u_1u_2^2)-\rho\beta_2(\sigma u_0^2u_2-\lambda\beta_3^2u_1^2u_2)=\sigma. \tag{24}$$

This shows that the octavic curve in which the quartic and quadric surfaces meet consists of the double conic, and the quartic on the double cone. The cubic (24) is in form similar to the map equations of plane sections, but differs from them in that there is not a linear relation existing between it and the four fundamental cubics.

GROUP PROPERTIES OF TRANSFORMATIONS ASSOCIATED WITH THE SURFACE.

§§ 12–13.

§ 12. *The Group of Correlations on the Surface.*

The four polarities (§ 3) and three known collineations C which leave the surface unaltered, constitute a group, G_8.

$$\begin{array}{ccc}
C_1. & C_2. & C_3. \\
\rho' x_0'=x_2 & \rho' x_0'=x_0 & \rho' x_0'=x_2 \\
\rho' x_1'=x_1 & \rho' x_1'=x_3 & \rho' x_1'=x_3 \\
\rho' x_2'=x_0 & \rho' x_2'=x_2 & \rho' x_2'=x_0 \\
\rho' x_3'=x_3 & \rho' x_3'=x_1 & \rho' x_3'=x_1
\end{array}$$

Forming the polars of a point y as to the quadrics Q (7), we have

$$\left.\begin{aligned}
P_1&\equiv m(ky_0+y_2)x_0+\mu n(ly_1+y_3)x_1+m(y_0+ky_2)x_2+\mu n(y_1+ly_3)x_3,\\
lP_2&\equiv m(ky_0+y_2)x_0+\mu n(y_1+ly_3)x_1+m(y_0+ky_2)x_2+\mu n(ly_1+y_3)x_3,\\
kP_3&\equiv m(y_0+ky_2)x_0+\mu n(ly_1+y_3)x_1+m(ky_0+y_2)x_2+\mu n(y_1+ly_3)x_3,\\
lkP_4&\equiv m(y_0+ky_2)x_0+\mu n(y_1+ly_3)x_1+m(ky_0+y_2)x_2+\mu n(ly_1+y_3)x_3.
\end{aligned}\right\} \tag{25}$$

If we transform any one of these polars by any one of the three collineations, the result is the equation of another polar. For example, $C_1P_1\equiv P_3$. To operate with one collineation on another gives the third, as $C_1C_3\equiv C_2$. As the result of carrying out these transformations in all possible ways, we have forms from which we observe the following facts:

1. Any collineation or any polarity operating on itself gives identity.
2. The product of two unlike collineations gives the third, and of two unlike polarities gives a collineation.
3. Each collineation and each polarity is its own inverse.
4. The products are commutative.

All requirements for a group are thus met and we may state our result in a

THEOREM. *The four polarities P and the three collineations C, with identity, form a group* G_8.

This group is defined by the equations

$$\begin{array}{cc} C_m^2=1 & P_i^2=1 \\ C_m=C_m^{-1} & P_i=P_i^{-1} \\ C_mC_k=C_l & P_iP_j=C_m \\ C_mC_k=C_kC_m & P_iP_j=P_jP_i \\ (m,\ k,\ l=1,\ 2,\ 3;\ m\neq k\neq l) & (i,\ j=1,\ 2,\ 3,\ 4;\ i\neq j) \end{array}$$

The generating elements are the collineations C_1 and C_2, and a polarity. The collineations form the only subgroup, which is a "four-group."

The multiplication table of the group is

	1	C_1	C_2	C_3	P_1	P_2	P_3	P_4
1	1	C_1	C_2	C_3	P_1	P_2	P_3	P_4
C_1	C_1	1	C_3	C_2	P_2	P_1	P_4	P_3
C_2	C_2	C_3	1	C_1	P_3	P_4	P_1	P_2
C_3	C_3	C_2	C_1	1	P_4	P_3	P_2	P_1
P_1	P_1	P_2	P_3	P_4	1	C_1	C_2	C_3
P_2	P_2	P_1	P_4	P_3	C_1	1	C_3	C_2
P_3	P_3	P_4	P_1	P_2	C_2	C_3	1	C_1
P_4	P_4	P_3	P_2	P_1	C_3	C_2	C_1	1

(26)

Any equation symmetric in x_0, x_2 and x_1, x_3 is invariant under the subgroup of collineations. Such forms are the equations of the planes $(x_0\pm x_2)+k(x_1\pm x_3)=0$, the quadric $x_0x_2+kx_1x_3=0$ and the surface itself. Invariant lines are given by

$$(1)\ \begin{array}{c} x_0=0 \\ x_2=0 \end{array} \qquad (2)\ \begin{array}{c} x_1=0 \\ x_3=0 \end{array} \qquad (3)\ \begin{array}{c} x_0+x_2=0 \\ x_1+x_3=0 \end{array} \qquad (4)\ \begin{array}{c} x_0-x_2=0 \\ x_1-x_3=0 \end{array}$$

The first three are invariant in that they are sent into themselves by G_4. The points of the last are absolutely invariant. These four lines, moreover, meet in four points and form two pairs of opposite generators on $x_0x_1-x_2x_3=0$. The polar as to the quadrics Q of a point on one generator of a pair is a plane on the other. We may then say that these lines are also invariant under G_8.

The 1, 1 correspondence between the points on (4) and the planes on (3) also arises from the operations of the group G_8 on points of the surface. By the collineations C, a point $(y_0,\ y_1,\ y_2,\ y_3)$ goes into

$$(y_2,\ y_1,\ y_0,\ y_3),\quad (y_0,\ y_3,\ y_2,\ y_1),\quad (y_2,\ y_3,\ y_0,\ y_1).$$

We call these four points a Y set. By the four polarities Q the same point goes into four planes P (25), the points of contact of which with the surface we call an X set. The planes P are found to meet in a point y' and the points Y to lie on a plane P', the polar of y' as to all four quadrics. The point is on the fixed line

$$x_0 - x_2 = 0, \quad x_1 - x_3 = 0;$$

the plane is on the other fixed line

$$x_0 + x_2 = 0, \quad x_1 + x_3 = 0.$$

By the properties of the group it is seen that the eight points X and Y form a closed set of which any one determines the remaining seven, and that the relation between the points X and their polar planes is in all respects the same as that between the points Y and their polar planes. Thus, if we start with a point X, we obtain by the polarities four planes touching the surface at points Y and meeting in a point which is the pole as to the quadrics Q of the plane which contains the points X. This new point and plane are found to lie respectively on the lines

$$x_0 - x_2 = 0, \quad x_1 - x_3 = 0 \quad \text{and} \quad x_0 + x_2 = 0, \quad x_1 + x_3 = 0.$$

The pairs of points, arising thus on the line

$$x_0 - x_2 = 0, \quad x_1 - x_3 = 0,$$

from every point on the surface, form an involution along the line, the fixed points of which are the two at which the four quadrics Q have double contact. The corresponding pairs of planes lie on the line

$$x_0 + x_2 = 0, \quad x_1 + x_3 = 0$$

also form an involution, the fixed planes of which are the two common tangent planes of the four quadrics at these points of contact.

We have thus the conclusion that the eight points and eight planes arising from the operations of the group on any point of the surface fall into two sets of four. Each set of four planes meets in a point on one line fixed under the group, while each set of four points lies in a plane passing through a second fixed line. The intersection of the planes tangent at one set of four points is the polar point as to the quadrics of the plane containing the four other points.

This group of correlations G_8 under which the surface is self-dual, is the only such group. If another group existed, it would be formed by adding to some new collineation group the known correlations. Applying the general collineation

$$\rho x_i' = \sum_{k=1}^{k=4} a_i x_k$$

to the equation of the surface, with the restriction that the surface remain unaltered, we find that the only solutions possible are the four transformations given by the collineation subgroup of G_8. The correlation group is, therefore, unique.

§ 13. *The Group of Cremona Transformations in the Mapping Plane.*

We have seen that the transformation of polarity as to the fundamental quadrics Q gives for any point y on the surface a plane tangent to the surface. The point of contact x is a double point on the section by the plane. Hence, the map equation of this plane is a rational cubic with double point z. The point y maps into some point u of the mapping plane. Thus, for a pair of points y and x, arising from a correlation on the surface, we have a pair u and z in the plane. We now proceed to derive a transformation, which applied to u, the map of a given point y, gives the point z corresponding to x.

The coordinates of the double point z in a curve mapping a tangent plane of the surface may be found by means of the combinant

$$f_0(J_{123})_{u=z}+f_1(J_{023})_{u=z}+f_2(J_{013})_{u=z}+f_3(J_{012})_{u=z}=0. \tag{27}$$

The quantities f_i are the four cubics (18), the quantities J_i are each the Jacobian of three of the cubics. Since the Jacobian of a net passes through the double points of the net, a curve given by (27) has, for a particular value of z, a double point at z. Let us set this equation identically equal to the map equation of a tangent plane of the surface, and equate coefficients of terms in u. We obtain four equations in y and z. But since y may be expressed as a function of u, our equations give z in terms of u. Solving, we have corresponding to the four polar transformations of a point y on the surface, four sets of relations between z and u in the plane.

$$\begin{aligned}
\Pi_1:\ & z_0=+\sqrt{\lambda}\beta_2\beta_3(nmu_0^2+2ln\beta_3u_0u_1+2km\beta_2u_0u_2+4kl\beta_2\beta_3u_1u_2),\\
& z_1=-\sqrt{\sigma}l\beta_2(nmu_0^2+2km\beta_2u_0u_2+2n\beta_3u_0u_1+4k\beta_2\beta_3u_1u_2),\\
& z_2=-\sqrt{-k\rho}\beta_3(nmu_0^2+2ln\beta_3u_0u_1+2m\beta_2u_0u_2+4l\beta_2\beta_3u_1u_2)\,;\\
\Pi_2:\ & z_0=+\sqrt{\lambda l}\,\beta_2\beta_3(nmu_0^2+2km\beta_2u_0u_2+2n\beta_3u_0u_1+4k\beta_2\beta_3u_1u_2),\\
& z_1=-\sqrt{\sigma}|\beta_2(nmu_0^2+2km\beta_2u_0u_2+2ln\beta_3u_0u_1+4kl\beta_2\beta_3u_1u_2),\\
& z_2=-\sqrt{-kl\rho}\beta_3(nmu_0^2+2m\beta_2u_0u_2+2n\beta_3u_0u_1+4\beta_2\beta_3u_1u_2)\,;\\
\Pi_3:\ & z_0=+\sqrt{\lambda k}\beta_2\beta_3(nmu_0^2+2\beta_2mu_0u_2+2ln\beta_3u_0u_1+4l\beta_2\beta_3u_1u_2),\\
& z_1=-\sqrt{kl\sigma}\beta_2(nmu_0^2+2m\beta_2u_0u_2+2n\beta_3u_0u_1+4\beta_2\beta_3u_1u_2),\\
& z_2=-\sqrt{-\rho}\,\beta_3(nmu_0^2+2km\beta_2u_0u_2+2ln\beta_3u_0u_1+4kl\beta_2\beta_3u_1u_2)\,;\\
\Pi_4:\ & z_0=+\sqrt{\lambda kl}\,\beta_2\beta_3(nmu_0^2+2m\beta_2u_0u_2+2n\beta_3u_0u_1+4\beta_2\beta_3u_1u_2),\\
& z_1=-\sqrt{k\sigma}\,\beta_2(nmu_0^2+2m\beta_2u_0u_2+2lm\beta_3u_0u_1+4l\beta_2\beta_3u_1u_2),\\
& z_2=-\sqrt{-l\rho}\,\beta_3(nmu_0^2+2km\beta_2u_0u_2+2n\beta_3u_0u_1+4k\beta_2\beta_3u_1u_2).
\end{aligned}$$

These transformations are involutory. We have then four birational transformations which set up a 1—1 correspondence between points u and z. Hence, they are Cremona transformations.

Such a transformation sends the ∞^2 lines of the plane into a net of conics on the three singular points of the transformation. Each quadratic expression for z factors into the maps of two tropes on the surface, and from its form is seen to pass through $(0,1,0)$ and $(0,0,1)$. Hence, the singular points for each transformation Π are these two reference points and the map of one pinch point. The singular lines of each transformation, given by the Jacobian of the quadratics in u, are $u_0=0$ and the maps of the two tropes which pass through this pinch point. The four points on the surface in which it is cut by the line of fixed points

$$x_0-x_2=0,\quad x_1-x_3=0$$

map into four points invariant under all the transformations Π.

If we apply these transformations to a point u, we obtain four points z. If we apply the same transformations to the point z, we obtain identity and three new points z'.

$$\begin{aligned}
&\mathrm{T}_1:\ z'_0=\lambda\beta_3^2u_0u_1; && z'_1=\nu u_0^2; && z'_2=\lambda\beta_3^2u_1u_2;\\
&\mathrm{T}_2:\ z'_0=\lambda\beta_2^2u_0u_2; && z'_1=\lambda\beta_2^2u_1u_2; && z'_2=-\rho u_0^2;\\
&\mathrm{T}_3:\ z'_0=\lambda\beta_2^2\beta_3^2u_1u_2; && z'_1=\sigma\beta_2^2u_0u_2; && z'_2=-\rho\beta_3^2u_0u_1.
\end{aligned}$$

The singular elements are the points and lines of the reference triangle. The fixed elements are the four points invariant under Π. These transformations T correspond to the collineations on the surface in the same sense that the transformations Π correspond to the polarities. The seven elements Π and T with identity form an abelian G_8 like in all particulars to the correlation group of the surface (26).

We have thus shown that the correspondence on the surface, arising from the correlation group, between a point y and a tangent plane with contact x, gives rise in the mapping plane to a group of Cremona transformations which show the same correspondence between the maps of the points y and x.